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PORTFOLIO CONSTRUCTION BY VOLATILITY FORECASTS: DOES THE COVARIANCE STRUCTURE MATTER?

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1. Introduction

MARKOWITZ (1952) proposed the expected return (mean) and the variance of the return of a portfolio of assets as criteria for optimal portfolio selection. He showed that the expected return of the portfolio is a weighted average of the expected returns of individual securities and the variance of the return of the portfolio is a particular function of the variances of, and the covariances between, securities and their weights in the portfolio. The performance of optimal mean-variance (MV) portfolios depends on the quality of the forecasts of the first two moments, i.e. the future mean returns and the variance matrix. CHOPRA and ZIEMBA (1993) have shown that estimation errors in the predicted returns are highly influential for the portfolio performance. Errors in variances and covariances are less important.
POJARLIEV and POLASEK (2001) (PP henceforth) have found in an empirical analysis that the weights of a global minimum variance (GMV) portfolio are very sensitive with respect to the input, i.e. the predicted variance matrix. Therefore the structure of the variance matrix and the selection of the appropriate volatility models will be important for the portfolio weights and determines the overall portfolio performance. CHAN et al. (1999) evaluate the performance of different multi-factor models for the covariance structure of stock returns. They conclude that a few factors capture the general covariance structure and that adding more factors does not improve the forecasts.
In practice, the portfolio construction is a process divided into two parts: the stock picking and the weights selection for the different stocks. Some portfolio managers are trying to track the benchmark by selecting a subset of stocks in order to reduce the transaction costs. Nevertheless, the number of assets in actively managed portfolios is usually high, which leads to dimensionality problems in the prediction of the variance matrix of the assets included in the portfolio. FLAVIN and WICKENS (2000) propose the use of multivariate GARCH model for

tactical asset allocation, i.e. the weights selection for stocks, long-term bonds, short-term bonds and cash.

A multivariate (classical) time series approach for portfolio construction will become numerically intractable for higher dimensional portfolios. LEDOIT et al. (2002) propose an alternative estimation method that is numerically feasible for large-scale portfolios. Recently, ENGLE (2000) and ENGLE and SHEPPARD (2001) proposed a new class of multivariate GARCH models (dynamic conditional correlation multivariate GARCH) capable to estimate large covariance matrices. In this paper we pursue a different approach: we look for patterns in the covariance matrix and we restrict the forecasting of the volatilities to the important elements of the covariance matrix. A limiting case of this approach is to use a diagonal structure for the variance matrix; this simplifies the prediction process and stabilizes the portfolio weights of a continuous updating process.

The computational effort is however still quite large and the application will be therefore limited to small size portfolios (asset allocations or counties bets). But what is the impact of such information losses (e.g. by setting covariances equal to zero) on the portfolio performance? We explore portfolios based on a diagonal structure for the variance matrix (of the assets which are included in the portfolio) and we evaluate this strategy by comparing the 'diagonal portfolio' with a portfolio based on the full variance matrix. Furthermore, we compare diagonal portfolio strategies based on univariate and multivariate GARCH models and we measure the 'value added' of this modelling approach by the relative differences of the SHARPE ratios.

Our main contribution to the existing approaches is that we show that using multivariate model for volatility forecasts lead to much better results than univariate models when used for portfolio construction. An interesting result is that the influence of volatility forecasts of models where zero covariances are assumed on the portfolio performance is very small and even may lead to better performance after transaction costs. The paper is organized as follows:

Section 2 describes the data set and the volatility forecasts. Section 3 presents the methodology for constructing the different portfolios and compares the performance. Some concluding remarks are given in the final section. A mathematical appendix describes the applied volatility models.

2. Volatility Forecasts

The volatility prediction of portfolio assets is one of the key factors for modern portfolio selection problems. Furthermore, it plays a significant role in derivative pricing. Many statistical models have been proposed to describe the behaviour of the volatility of the stock markets, like rolling variance estimates, ARCH models and non-parametric methods. Empirical research has uncovered a series of stylized facts about the volatility of the stock markets and a recent survey is given in ENGLE and PATTON (2000).

There exists a wide-spread consensus that volatility processes exhibits persistence (i.e. the conditional return variances have a lasting effect on the annualized variance over many periods ahead); these conditional variances are assumed to be 'mean reverting' such that there is a certain level of volatility to which the conditional variances will return after an innovation shock. We investigate the volatility of the daily returns of the MSCI North America, MSCI Europe and MSCI Pacific indices from 1st May 1995 until 3rd April 2000. The first 800 observations (from 1st May 1995 until 22nd May 1998) are used as a "training" sample for the model selection and the rest for the out-of-sample evaluation.

Table 1: Annualized Standard Deviations (SD).

MSCI Region	Pacific	Europe	North America
1998 annualized SD	28.44%	20.65%	19.40%
Full Sample annualized SD	20.73%	14.47%	16.97%

Table 1 contains the annualized standard deviations (SD) of the daily returns of the three MSCI indices.
Table 1 shows that the year 1998 was an exceptional volatile year: The annualized SD in 1998 is much higher for the three MSCI indices in comparison to the annualized SD over the last five years (1st May 1995 until 3rd April 2000). Presumably the Asia crisis in 1997/98 was responsible for the higher volatility in the stock markets in 1998. This leads us to consider an asymmetric GARCH model to forecast the volatilities of the three MSCI indices.

2.1 Univariate Modelling

We estimate an AGARCH(1,1) model for the daily returns of the three MSCI indices using the last 800 observations (approximately 3 years) of our time horizon. For the estimation of the (non-linear) model we use the BHHH algorithm of BERNDT et al. (1974).
The AGARCH (1,1) models for the daily returns of the MSCI regions (using the last 800 observations of our data set, from 1st May 1995 until 22 May 1998) are estimated as follows:

1. MSCI Pacific index

$$\hat{\sigma}_t^2 = 10^{-7}6.6 + (0.017 + 0.05 S_{t-1})\varepsilon_{t-1}^2 + 0.95\sigma_{t-1}^2$$

(t - val.) (1.87) (1.54) (3.58) (78.62)

2. MSCI Europe index

$$\hat{\sigma}_t^2 = 10^{-7}4.1 + (0.043 + 0.02 S_{t-1})\varepsilon_{t-1}^2 + 0.94\sigma_{t-1}^2$$

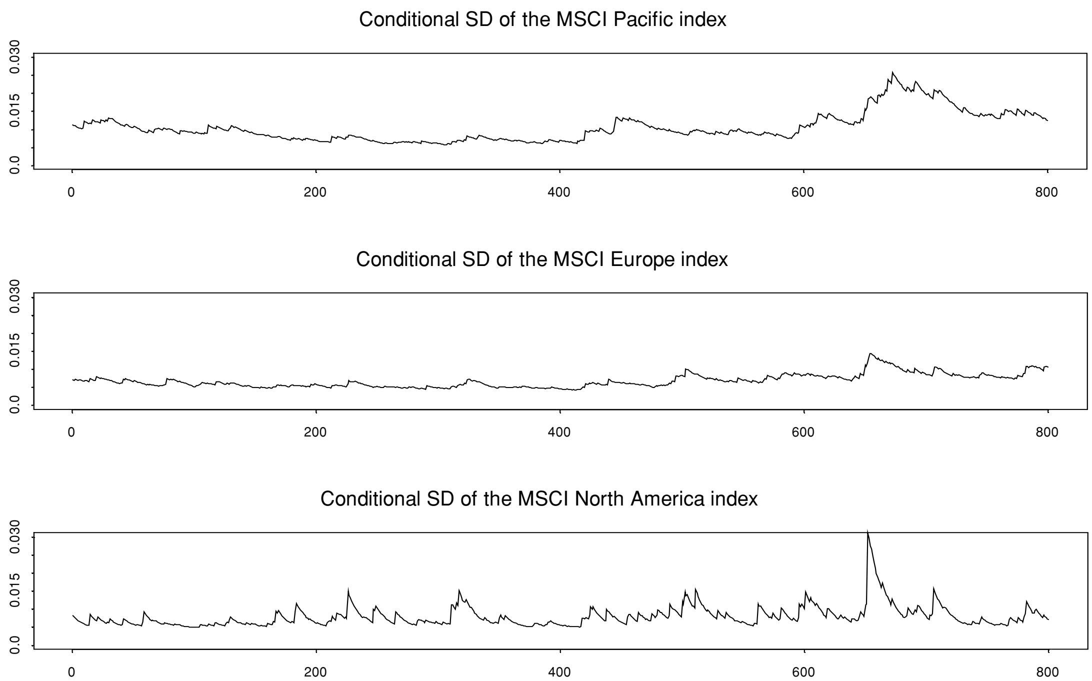
(t - val.) (1.30) (1.96) (0.78) (52.33)

3. MSCI North America index

$$\hat{\sigma}_t^2 = 10^{-7}3.3 + (0.003 + 0.18 S_{t-1})\varepsilon_{t-1}^2 + 0.86\sigma_{t-1}^2$$

(t - val.) (4.23) (0.19) (7.27) (44.21)

Figure 1: Conditional SD of the three MSCI Indices from 1st May 1995 until 22nd May 1998



The comparison of the regional models exhibits several interesting results: First note that the asymmetry parameter can be estimated significantly for the Pacific and the North American region but not for Europe (for the period 1995-1998). The asymmetry coefficient γ_1 is the largest for the MSCI North America index but the estimated α_1 and β_1 parameters are the smallest. This shows that the volatility of the daily returns follows different patterns in the 3 regions, and Europe was exposed to a more balanced volatility (and risk) process than the rest of the world at the end of 1997 when there was the volatility shock induced by the Asian financial market crisis. This can be seen graphically from Figure 1, which plots the conditional SD of the returns of the three MSCI indices and illustrates the higher volatility at the end of 1997 (between observations 650 and 750).

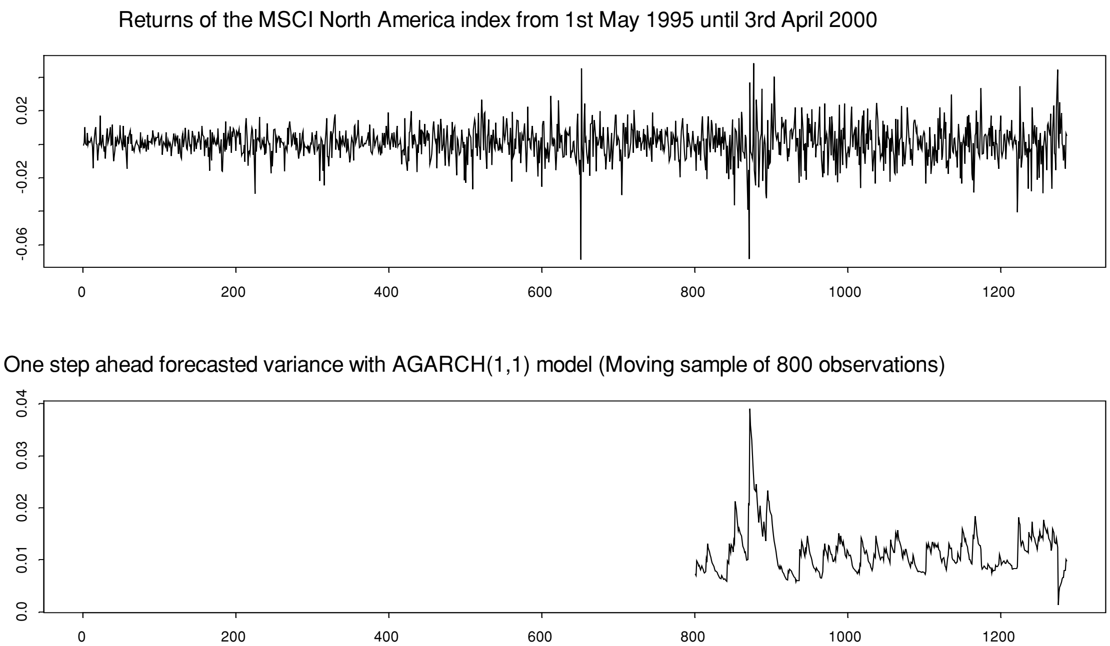
How does this asymmetry affect the persistence behavior? The sum of the estimated ARCH coefficients $\alpha_1 + \beta_1$ is less than 1 for positive shocks but $\alpha_1 + \beta_1 + \gamma_1$ is larger than 1 for negative shocks. This means that negative residuals at time t tend to lead to a higher conditional volatility in the period $t+1$.

The goal of our volatility models is to use the predictions of the GARCH model for the portfolio construction. Using the AGARCH model we calculate the conditional volatility forecasts for the next trading day (25 May 1998) by:

$$\hat{\sigma}_{t+1}^2 = \hat{\alpha}_0 + (\hat{\alpha}_1 + \hat{\gamma}_1 S_t)\varepsilon_t^2 + \hat{\beta}_1 \sigma_t^2. \qquad (1)$$

This procedure is repeated 486 times using a rolling sample of 800 observations and re-estimating the AGARCH(1,1) models for each

Figure 2: The out-of-Sample Volatility Forecasts



MSCI region. This was a very computationally intensive task. We have used the S+GARCH module of Splus. The 486 re-estimations of each model have consumed about 2 hours. However it took more than 20 hours to re-estimate 486 times the multivariate GARCH model described in the next section. Thus, we forecast the conditional variances of the returns of the MSCI Pacific, MSCI Europe and MSCI North America indices from 25th May 1998 until 3rd April 2000. Figure 2 plots as an example the predicted conditional variance ($\hat{\sigma}^2_{t+1}$) of the daily returns of the MSCI North America index.

2.2 Multivariate Modelling

We have used the multivariate BEKK model of ENGLE and KRONER (1995) to forecast the variance matrix. For the order selection of the multivariate GARCH process we have calculated the *AIC* or *BIC*[1] values in Table 2. Using the *AIC* criterion, we select a BEKK(2,1) model for the three MSCI indices. The BEKK(2,1) model was preferred to the BEKK(1,1) model because the parameter estimates of the lag 2 matrix in equation (14) contain significant coefficients on the main diagonal, i.e. will contribute to the forecasts of the conditional variances. Note that the return vector r_t in equation (15) is specified as $r_t = (r_t^P, r_t^E, r_t^A)'$ where the letters *P*, *E* and *A* stand for the MSCI Pacific, MSCI Europe and MSCI North America indices, respectively. We have used the geographical order (from east to west) and the closing values of the indices to compute the returns.

Table 2: AIC and BIC Values for Different Lag Orders of the BEKK Model.

	AIC	BIC
BEKK(1,1)	–16388	–16135*
BEKK(2,1)	–16400*	–16035
BEKK(2,2)	–16323	–15901
BEKK(3,2)	–16164	–15658

The Star (*) Denotes the Smallest Values.

Thus, the variance matrix forecast is obtained from

$$\hat{H}_{t+1} = \hat{A}_0\hat{A}_0' + \sum_{i=1}^{p}\hat{A}_i E_t(\varepsilon_{t+1-i}\varepsilon_{t+1-i}')\hat{A}_i' + \sum_{i=1}^{q}\hat{B}_i H_{t+1-i}\hat{B}_i'. \quad (2)$$

where E_t is the conditional expectation operator. The estimated coefficients of the BEKK(2,1) model for the period of 800 observations are given by (t-values are in parenthesis):

$$\hat{\mu} = \begin{pmatrix} -0.00029(-0.90) \\ 0.00083(3.52) \\ 0.00073(2.49) \end{pmatrix} \quad (3)$$

$$\hat{A}_0 = \begin{pmatrix} 0.0004(0.30) & \% & \% \\ 0.0014(0.30) & -0.0016(-0.23) & \% \\ 0.0002(0.00) & 0.0012(0.00) & 0.0006(0.00) \end{pmatrix} \quad (4)$$

$$\hat{A}_1 = \begin{pmatrix} 0.22(5.95) & 0.08(2.19) & -0.01(-0.37) \\ 0.02(0.33) & 0.03(0.58) & -0.11(-1.77) \\ -0.18(-4.09) & -0.13(-4.48) & 0.12(4.05) \end{pmatrix} \quad (5)$$

$$\hat{A}_2 = \begin{pmatrix} -0.08(-1.35) & -0.10(-2.79) & -0.03(-0.90) \\ 0.11(1.49) & 0.22(4.93) & 0.09(1.57) \\ 0.04(0.67) & -0.01(-0.32) & 0.25(4.67) \end{pmatrix} \quad (6)$$

$$\hat{B} = \begin{pmatrix} 0.96(111.90) & -0.01(-1.22) & 0.00(0.13) \\ -0.06(-1.45) & 0.90(27.96) & 0.03(0.84) \\ 0.03(0.70) & 0.05(1.82) & 0.90(25.60) \end{pmatrix} \quad (7)$$

Note that all non-diagonal elements of the matrix $\hat{B}$ in (7) are not significant and that only a few non-diagonal elements of the matrices A_1 and A_2 are estimated significantly.

As an overall diagnostic check we have calculated the residual autocorrelation function (ACF) of the sum of squared returns, which corresponds to the squared norm of the return vector. The ACF of the squared norm of the returns of the three MSCI indices exhibits significant correlation and has motivated the use of a multivariate GARCH model. The second panel of Figure 3 shows the ACF of the squared norm of the standardized residuals for the fitted BEKK(2,1) model. Except for lag 1 all significant autocorrelations stay within the asymptotic (+/–2) standard error bounds (dotted lines).

With the estimated parameters of the BEKK model in (2) and (11) we calculate the one-step-ahead forecasts for the next trading day (25 May 1998). As for the AGARCH model, we repeat this procedure 486 times using a rolling sample of 800 observations where we re-estimate the BEKK(2,1) model each time. Figure (4) shows these forecasts for the variance of the MSCI Europe index in the lower panel (the second diagonal element of the forecasted variance matrix $\hat{H}_{t+1}$).

Figure 3: The ACF of the "Squared Norm Returns" and the Squared Norm of the Standardized Residuals

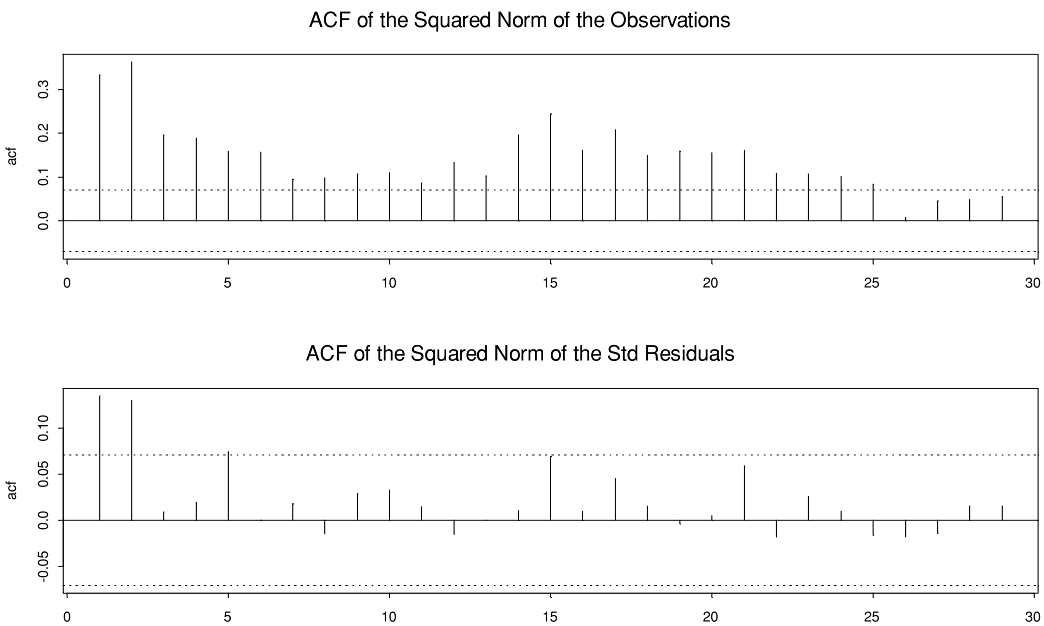
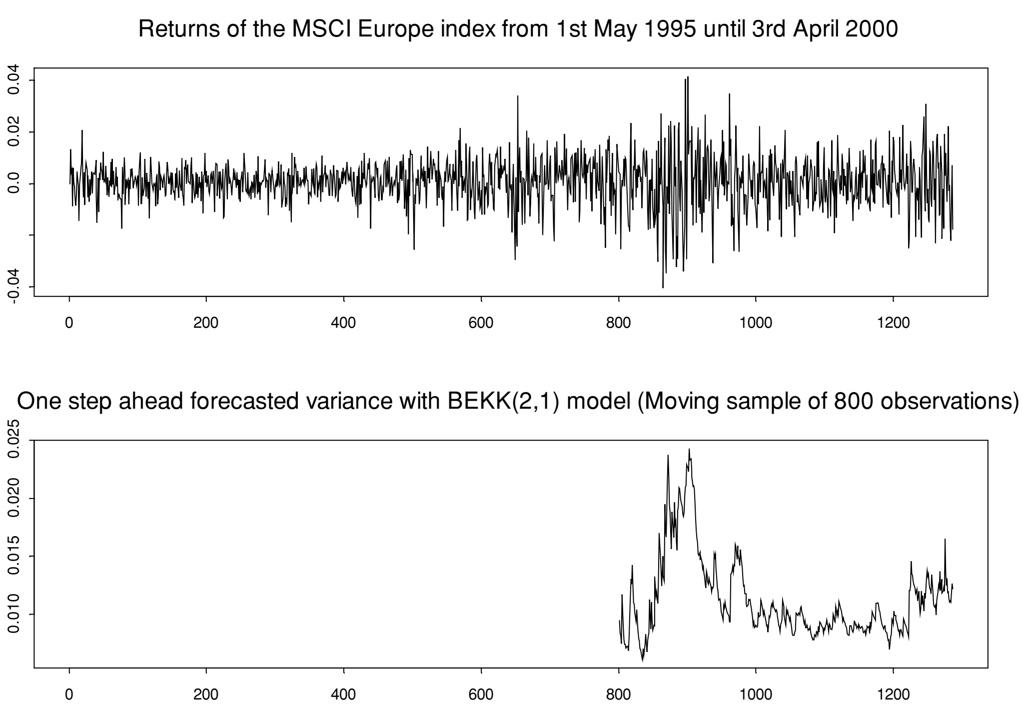


Figure 4: Out-of-Sample Volatility Prediction for the 25th May 1998 and the Following 485 Days



3. Portfolio Construction and Volatility Forecasts

Assuming a portfolio that consists of the 3 MSCI indices, we have to compute the weights of the portfolio assets for each index. The weights of the *global minimum variance* (GMV) portfolio ω_i depend only on the predicted variance matrix $\hat{H}_{t+1}$ (see CAMPBELL et. al. (1997)). Thus, by predicting the variance matrix for the next time point t+1 we can compute the optimal (expected) weights of a GMV portfolio. We investigate the following questions: First, what is the 'portfolio gain' when we go from univariate forecasts to multivariate forecasts, i.e. can we quantify the value of the information gained by multivariate time series forecasts? Second, how do portfolio weights depend on a changing covariance structure?

1. The Univariate Diagonal (UD) Portfolio
We assume a diagonal variance matrix for the three MSCI indices to compute the optimal weights of portfolio one. The weights of the univariate diagonal portfolio are given by the share of an asset precision (inverse variance) over the sum of all precisions:

$$\omega_{t,i} = \frac{\hat{\sigma}^{-2}_{t+1,i}}{\sum_{j=1}^{3} \hat{\sigma}^{-2}_{t+1,j}}, \quad i = 1,2,3 \tag{8}$$

where $\hat{\sigma}^2_{t+1,i}$ is the predicted conditional variance of the daily returns of the i^{th} MSCI index (for i = North America, Europe and the Pacific region) by the AGARCH(1,1) model as described in the previous section.

2. The Multivariate Diagonal (MD) Portfolio
We compute the weights for portfolio two using again equation (8), but the forecasted volatilities are the diagonal elements of the predicted variance matrix $\hat{H}$ of the BEKK model. Figure 5 shows the weights of the MD portfolio.

3. The Global Minimum Variance (GMV) Portfolio
The GMV portfolio is based on the full variance matrix forecasts $\hat{H}$ from the BEKK model (described also in PP (2001)) and the weights are computed as following:

$$\omega_t = \frac{\hat{H}^{-1}_{t+1}\iota}{\iota'\hat{H}^{-1}_{t+1}\iota}. \tag{9}$$

where ι is the vector of ones.

Additionally, we have also calculated the returns for the optimal SHARPE ratio portfolio, i.e. a portfolio with a riskless return (see CAMPBELL et al. (1997) p.88). This portfolio produced in all our calculations the worst portfolio performance and therefore we have omitted it for the comparison. The bad performance of this portfolio has at least two reasons: Firstly, as mentioned before, the prediction errors for the future returns can be very damaging for the portfolio performance and secondly, although daily asset returns are rather unpredictable, return volatilities are on average better predictable. Ironically the *maximum* SHARPE ratio portfolio has a lower SHARPE ratio than the *global minimum variance* portfolio. Since the GMV portfolio depends only on the predicted variance matrix it does not depend on return forecasts.
Figure 6 shows the dependence of the portfolio weights on the predicted variances. The first panel compares the variance forecasts of the AGARCH models and of the BEKK model for the returns of the MSCI Pacific index. The second panel plots the resulting weights of the univariate diagonal (UD) portfolio and multivariate diagonal (MD) portfolio for the Pacific region computed by equation (9). We see that the weights fluctuate considerably since they are very sensitive to the input, i.e. the inverse of the predicted variance matrix. Therefore the selection of a reliable volatility model will mainly determine the portfolio performance.

Figure 5: The Weights of the MD Portfolio from 25th May 1998 until 3rd April 2000

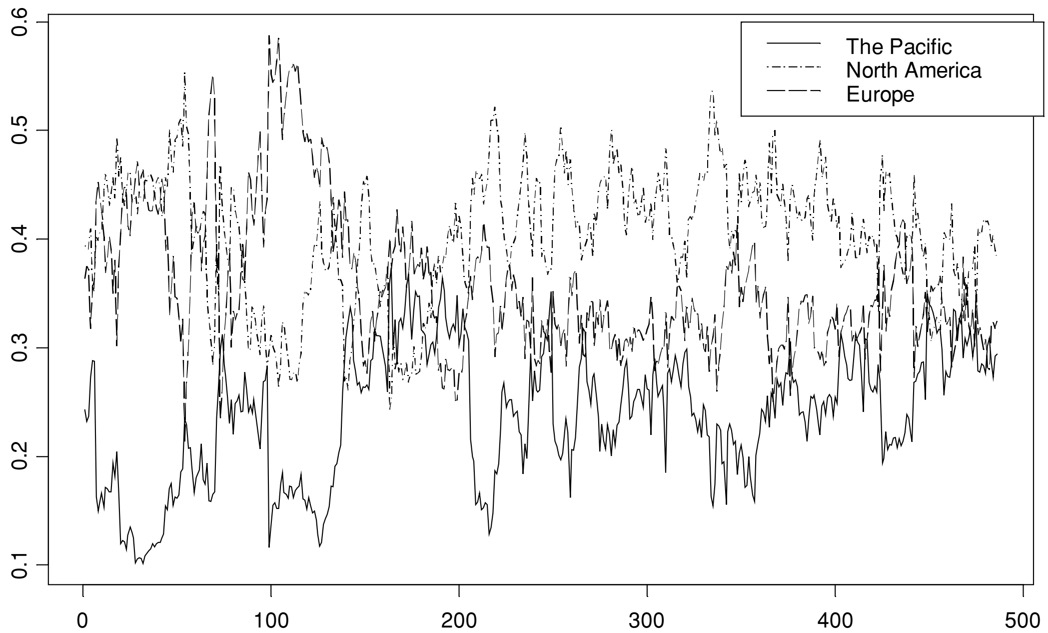
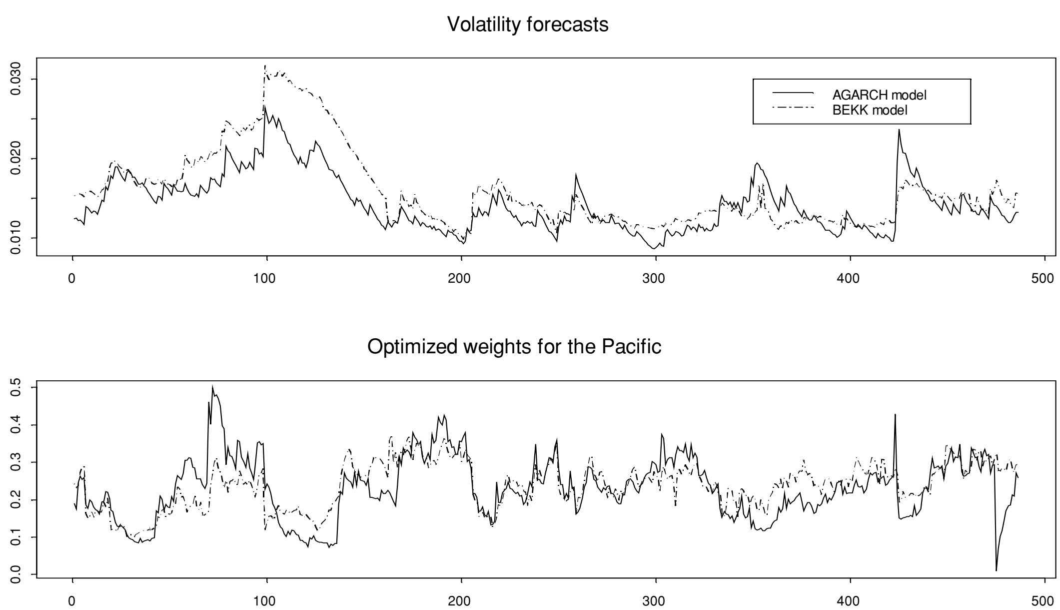


Figure 6: Portfolio Weights and Volatility Forecasts



3.1 Portfolio Evaluation

We compare the performance of the three portfolios for the period from 25th May 1998 until 3rd April 2000. As a benchmark return we assume the MSCI World index. Table 3 summarizes the cumulative returns, the standard deviations (SD) of the returns, the resulting SHARPE ratios and the annualized returns before and after transaction costs (TC). To keep transaction costs low, we allow for trading of futures, in which case transaction costs are pretty low. In order to investigate the effect of different transaction costs, we are calculating the results for two different assumptions: In case A transaction costs are 5 basis points (0.05%), in case B transaction costs are double as high, i.e. 10 basis points (0.1%). The daily transaction costs are then calculated by multiplying the daily portfolio turnover by 0.05% and by 0.1% respectively. The cumulative turnover for the GMV portfolio is 59.6 (or 5960%) for 486 days, which lead to 2.98% cumulative transaction costs. The cumulative turnover of 59.6 implies that the portfolio is totally rebalanced – on average – every 8 trading days.
An interesting result is that the use of a diagonal covariance structure leads to smaller turnover and therefore to smaller transaction costs. The UD portfolio has a cumulative turnover of 39.1, the MP portfolio of 27.4 (half the size of the turnover of the GMV portfolio). The reason is that the optimised portfolio weights are more stable, mainly because there is no new information regarding the covariance structure (covariances are assumed to be constant, i.e. equal to zero).
Although the SHARPE ratio is defined as the ratio of the *excess* portfolio return over volatility, many investment funds simply use the ratio of the annualised portfolio return and the annualised volatility. Therefore, we compute the SHARPE ratio as $SR_p = \mu_p/\sigma_p$ with μ_p being the annualised return and σ_p the annualised standard deviation of the portfolio. For the time series model and the volatility forecasts we have used log-returns, and for the portfolio comparison we used simple returns.
Comparing the results for the portfolio returns and their SDs are quite interesting. Without taking transaction costs into account, both the MD portfolio and the GMV portfolio have larger annualized average returns and smaller annualised SDs. However, statistical inference shows that the outperformance (the difference in the mean returns) is not statistically significant. A paired t-test[2] fails to reject a null hypothesis of equal mean returns at a 95% confidence level. This is not surprising since portfolio returns are by nature highly volatile, which means that we get only significant results if the difference between benchmark and active portfolio returns are huge, i.e. larger than the underlying SD, which is highly unlikely. The other possibility is that we need large number of data points. But as we can see from Table 3, the annualised SD (based on 250

Table 3: Performance Evaluation (25th May 1998 – 3rd April 2000, 486 Observations)

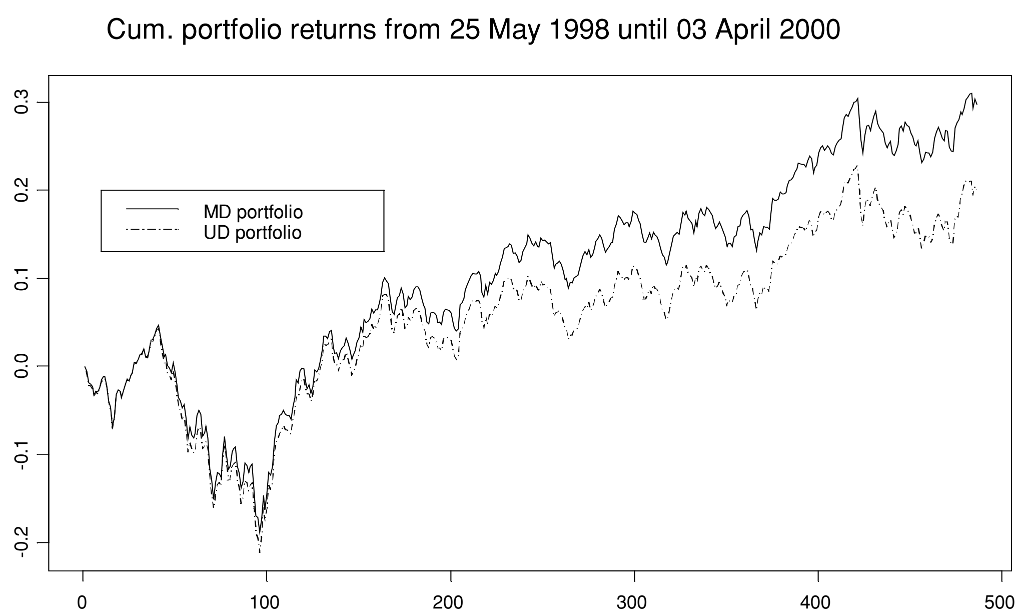
	UD portfolio	MD portfolio	GMV portfolio	benchmark
Returns	20.02%	29.63%	31.85%	27.46%
SD	20.77%	20.38%	20.62	21.11%
Annualized Returns	10.30%	15.24%	16.39%	14.13%
Annual. Returns after TC(A)	9.29%	14.53%	14.85%	
Annual. Returns after TC(B)	8.28%	13.82%	13.31%	
Annualized SD	14.90%	14.61%	14.80%	15.14%
Annualized SHARPE Ratio	0.691	1.043	1.11	0.933
Annual. SHARPE Ratio after TC(A)	0.623	0.995	1.00	0.933
Annual. SHARPE Ratio after TC(B)	0.556	0.946	0.899	0.933

observations) is only about 5–6% lower than the sample SD. Thus, it seems almost impossible to reach statistically significant results for active portfolio management. However, for practical purposes even small positive differences in annualised returns are important. Many active portfolio managers have been criticized of lying below the benchmark although those results are also not statistically significant. Therefore, we conclude, that although the outperformance is not statistically significant, any returns that lie above the benchmark are an indication that carefully chosen volatility forecasts can improve active portfolio management.

After transaction costs of 0.05% (case A) both the GMV and the MD portfolios still have larger SHARPE ratios than the benchmark. Assuming higher transaction costs (case B) leads to interesting results. In this case the MD portfolio has the highest SHARPE ratio (0.946). This shows that the value of covariance forecasts is too small to offset for transaction costs. Assuming zero covariances will therefore not only simplify the estimation procedure, but may even lead to better portfolio performance.

The univariate diagonal (UD) portfolio has the lowest cumulative returns (20.02%) and the lowest SHARPE value before and after transaction costs. The differences between the SHARPE ratios of the MD and UD portfolios can be used as a measure for the improvement (value added) through the multivariate modelling (the BEKK model).

Figure 7: Cumulative Returns for the Univariate UD and Multivariate MD Portfolios from 25 May 1998 until 3 April 2000



4. Conclusion

This paper has analyzed the performance of quantitative portfolio strategies which depends crucially on the ability of time series models to forecast variances and covariances. Our empirical results point in a clear direction: Portfolios based on multivariate GARCH volatility forecasts have higher SHARPE ratios than the benchmark (MSCI World) in the evaluation period. Even if the outperformance is not statistically significant, there is an indication that volatility modelling has added value.

The UD portfolio based on univariate GARCH volatility forecasts performs poor. This shows that multivariate models lead to better results than univariate models when implied for portfolio construction. The performance comparison between the MD portfolio and the 'full-information' GMV portfolio has shown that portfolios with an appropriate covariance structure can even outperform the 'full-information' portfolio after transaction costs. This indicates that restricting the covariance structure may in some cases improve the portfolio performance. We conclude that only those covariance components should be used for calculating optimal portfolio weights which are reliable, i.e. which are not too volatile in an updating prediction process and which can be predicted with sufficient accuracy. This implies that we should look for "constant patterns" in the covariance matrix of a portfolio problem. The updating process will then not lead to additional noise in the weights calculation and therefore will save transaction costs (stemming from unnecessary realignment of a portfolio). Such a feature will be important for the construction of quantitative portfolios. However, further research is needed to analyze the impact of patterns in the covariances for different portfolio settings, especially in large scale portfolios.

APPENDIX

1. The Asymmetric GARCH Model

The idea of the AGARCH model is that asymmetric behaviour of negative shocks are sources for additional risks. We specify an asymmetric GARCH model for the index returns and normally distributed errors

$$r_t | I_{t-1} \sim N[\mu, \sigma_t^2] \qquad (10)$$

or

$$r_t = \mu + \varepsilon_t, \quad t = 1,...,T \qquad (11)$$

where I_{t-1} is the information set until time $t - 1$. We assume a constant mean μ for the returns and for the errors ε_t a Gaussian distribution with mean zero and variance σ_t^2. The asymmetric GARCH model of orders p and q, i.e. an AGARCH(p,q) model, parameterizes the conditional variances by the following form (see e.g. GLOSTEN et al. 1993)

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} (\alpha_i + \gamma_i S_{t-i}) \varepsilon_{t-i}^2 + \sum_{i=1}^{q} \beta_i \sigma_{t-i}^2 \qquad (12)$$

where S_t is the dummy variable for the negative residuals and is defined as

$$S_t = \begin{cases} 1 & if \quad \varepsilon_t < 0 \\ 0 & if \quad \varepsilon_t \geq 0 \end{cases} \qquad (13)$$

2. The Multivariate GARCH (and BEKK) Model

Let $r_t = (r_t^1, ..., r_t^N)'$ be a N dimensional vector of returns at time t and we specify the following multivariate GARCH model

$$r_t = \mu + \varepsilon_t, \quad t = 1,...,T \qquad (14)$$

with

$$\varepsilon_t | I_{t-1} \sim N(0, H_t) \qquad (15)$$

where μ is a constant mean vector of dimension N and the heteroskedastic errors ε_t are *conditionally* on I_{t-1} multivariate normally distributed. Each element of H_t depends on p lagged values of squares and cross-products of ε_t^l, $l = 1,...,N$ and on q lagged values of H_t.
Defining $h_t = vecH_t$ as the vectorisation of a symmetric matrix and $\eta_t = vec(\varepsilon_t \varepsilon_t')$ then the multivariate GARCH(p,q) parameterization of the variance matrix can be written as

$$\begin{aligned} h_t &= a_0 + A_1 \eta_{t-1} + ... + \\ &\quad + A_p \eta_{t-p} + B_1 h_{t-1} + ... + B_q h_{t-q} \end{aligned} \qquad (16)$$

where a_0 is a $n \times 1$ vector with $n = N(N + 1)/2)$ and the A_i's and B_i's are $n \times n$ parameter matrices. This parameterization is also called *vec* representation. BOLLERSLEV et al. (1998) have proposed a *diagonal* representation, in which each element of the variance matrix $h_{jk,t}$ depends only on past variances and the past values of $\varepsilon_t^l \varepsilon_t^k$. This means that the conditional variances depend on past own variances and past squared residuals; likewise the covariances depend on past own covariances and cross products of residuals. In the *vec* representation the *diagonal* model is obtained by assuming a diagonal structure of the matrices A_i and B_i.
In both representations it is difficult to impose the condition of a positive definite variance matrix for the estimation procedure. ENGLE and KRONER (1995) propose the so-called BEKK representation that ensures the condition of a positive definite conditional variance matrix by a special matrix form. A BEKK(p,q) model parameterizes the variance matrix by the following way:

$$H_t = A_0 A_0' + \sum_{i=1}^{p} A_i (\varepsilon_{t-i} \varepsilon_{t-i}') A_i' + \sum_{i=1}^{q} B_i H_{t-i} B_i' \qquad (17)$$

FOOTNOTES

[1] The information criteria AIC (Akaike criterion) and BIC (Bayesian or Schwartz criterion) are given by AIC = log s^2_{ML} + $2k/n$ and BIC = logs^2_{ML} + $k\log n/n$ respectively, where is the ML estimate of the residual variance, k is the number of independent variables, and N is the number of observations.

[2] The paired t-test tests for a significant difference of the mean in dependent samples

data: x: Returns of the GMV portfolio, and
y: Returns of the MSCI World Index

Then the paired two-sample t-test yields a t = 0.1489 for df = 970 degrees of freedom, and the p-value = 0.4408 tells us that the difference in returns is not significant. (Note that we used a one side alternative hypothesis: true difference in means is greater than 0.)

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